

Bi-Lipschitz equivalent Alexandrov surfaces, II

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1 Basic definitions and statements

This paper is a continuation of the paper [BeBu]. Recall that a map $f : X \rightarrow Y$ of a metric space (X, d_X) in a metric space (Y, d_Y) is called bi-Lipschitz with a constant L (or L -bi-Lipschitz) if for every $x, y \in X$

$$L^{-1}d_X(x, y) \leq d_Y(f(x), f(y)) \leq Ld_X(x, y).$$

In this case, the spaces X, Y is called bi-Lipschitz equivalent (with constant L). In other words, two metric spaces are L -bi-Lipschitz equivalent if the Lipschitz distance $d_{Lip}(X, Y)$ is not greater, than $\lg L$.

Our readers supposed to be familiar with the basic notions of two dimensional manifolds of bounded total (integral) curvature theory. Its expositions can be found, for instance, in [AZ] and [Resh].

Hereafter the notion of Alexandrov surface means a *complete* two dimensional manifold of bounded curvature with a boundary; the boundary (which may be empty) is supposed to consist of a finite number of curves with finite variation of turn.

Notations: let M be an Alexandrov surface with metric d , ω be its curvature, which is a signed measure, ω^+ , ω^- be positive and negative parts of the curvature, and $\Omega = \omega^+ + \omega^-$ be variation of the curvature. For any Riemannian manifold M and a Borel set $E \in M$, $\omega^+(E) = \int_E K^+ dS$, $\omega^-(E) = \int_E K^- dS$, where K is Gaussian curvature.

A point p carrying curvature 2π and a boundary point carrying turn π are called peak points.

*The author was partly supported by grants RFBR 02-01-00090, SS-1914.2003, CRDF RM1-2381-ST-02, and Shapiro Foundation of Pennsylvania State Univ. (US).

We use notation $|xy|$, for the distance $d(x, y)$; by $s(\gamma)$ and $S(E)$ denote the length of a curve γ and the area of a set E , correspondingly. $\mathbf{D}(X, r)$ means the disk of radius r centered at X .

For positive numbers D, C, l, ϵ and integer χ , by $\mathfrak{M} = \mathfrak{M}(\chi, D, C, l, \epsilon)$ we denote the class of closed oriented Alexandrov surfaces M having Euler number χ and satisfying the following conditions:

- (i) $\text{diam } M \leq D$,
- (ii) $\omega^-(M) \leq C$,
- (iii) if the length of a simple closed curve is less than l , then the curve is the boundary of a disk $\mathbf{D} \subset M$ such that $\omega^+(\mathbf{D}) \leq 2\pi - \epsilon$.

It follows from (iii) that for every point $p \in M$ the condition $\omega(p) \leq 2\pi - \epsilon$ holds. Particularly, M has no peak points. Besides, the systolic constant for M is not less than l . (Recall that the systolic constant $\text{sys } M$ of a close surface M is the infimum of lengths of noncontractible curves in M).

Classes \mathfrak{M} are compact; the proof is standard, see Section 2.

The following theorem is the main result of the paper:

Theorem 1. *There exists a positive constant L , depending on χ, D, C, l, ϵ only such that $d_{Lip}(M_1, M_2) \leq L$ for any two Alexandrov surfaces $M_1, M_2 \in \mathfrak{M}$.*

Remark 1.

1. A similar theorem is also valid for nonoriented surfaces.
2. Theorem 1 is a generalization of Theorem 1 from [BeBu], but its proof is not independent of the latter one.
3. A generalization of Theorem 1 for Alexandrov surfaces with nonempty boundaries takes place. Naturally, we have to add boundary conditions in the description of classes \mathfrak{M}' of surfaces with boundaries. Namely, distances between boundary components have to be uniformly separated from zero, say by a number l . Besides, for every two boundary points x, y , the ratio of smaller boundary arc between x and y to the distance $d(x, y)$ also has to be uniformly separated from zero. The latter condition implies that there is no boundary point with turn greater than $\tau(q) = \pi - \epsilon'$ for some fixed $\epsilon' > 0$ (but we do not exclude points q with $\tau(q) = -\pi$). We always suppose boundaries to consist of a finite number of curves having bounded variation of turn (this condition can be weakened). We drop the precise formulation because it is a bit complicated.

In case of surfaces with boundary, the proof is basically the same as for closed surfaces. Also, it is sufficient to apply Theorem 1 to the doubling of a

surface with boundary because freedom in the choice of a bi-Lipschitz map allows to find it such that it moves boundaries one to the other.

Let T be an end; i.e., an Alexandrov surface homeomorphic to a closed disk with its center removed and such that $d(a, p_i) \rightarrow \infty$ as $i \rightarrow \infty$ for any sequence of points $p_i \in T$ whose images in the disk converge to its center. Here a is a fixed point. We call the quantity $v = -\tau(\gamma) - \omega(T)$ the growth speed of the end T . Here $\tau(\gamma)$ is the turn of the boundary γ of the end. From the Cohn-Vossen inequality it follows that $v \geq 0$. Note, that the growth speed of an end is positive if and only if the limit

$$v(T) = \lim_{i \rightarrow \infty} \frac{l(\gamma_i)}{d(a, p_i)},$$

where $l(\gamma_i)$ is the length of the shortest noncontractible loop with the vertex p_i . Under condition $\Omega(T) < \infty$, this limit is well-defined and is not greater than 2.

Every open (i.e., complete and equipped with an unbounded metric) finitely connected Alexandrov surface can be cut (for instance, by geodesic loops) onto a compact part M_c and ends T_i . Let us consider classes $\mathfrak{M}^* = \mathfrak{M}^*(g, C, l, \epsilon, v_0,)$ consisting of homeomorphic one to another Alexandrov surfaces M of genus g , satisfying the conditions (ii) - (iii) from the definition of class \mathfrak{M} and such that all ends have growth speeds not less, than the number v_0 (growth speed of an end does not depend on choice of a loop γ_i in its homotopy class). We will choose loops γ_i in such a way that ends T_i would satisfy the conditions: $\Omega(T_i) + \tau^+(\gamma_i) < 0,001$, where τ is the turn from the end side, and $\text{sys}(\text{doubl } M_c) \geq \text{sys } M_c$. Here $\text{doubl } M_c$ is the double of M_c . These conditions can definitely be satisfied if we choose loops far enough from some fixed point.

Let us denote $\tilde{\mathfrak{M}} = \tilde{\mathfrak{M}}(g, D, C, l, \epsilon, s, v_0)$ the subset of class \mathfrak{M}^* consisting of surfaces which can be decomposed onto a compact part M_c and ends T_i such that the conditions listed above hold true and, in addition,

$$\text{diam } M_c \leq D, \quad \text{length}(\gamma_i) \leq s.$$

It is clear that every surface of class \mathfrak{M}^* belongs to some class $\tilde{\mathfrak{M}}$. Now Corollary below follows immediately from Theorem 1 and Remark 2 from the paper [BeBu].

Corollary 1. *There exists a constant L_1 , depending on $g, D, C, l, \epsilon, s, v_0$ only such that all Alexandrov surfaces of class $\tilde{\mathfrak{M}}(g, D, C, l, \epsilon, s, v_0)$ are L_1 -bi-Lipschitz equivalent.*

The author thanks A. Belenkiy and V. Zalgaller whose advises help to simplify some proofs.

A sketch of the proof of Theorem 1

By $L(M, N)$ we denote infimum of Lipschitz constants for bi-Lipschitz maps $M \rightarrow N$, where $M, N \in \mathfrak{M}(D, C, \chi, l, \epsilon)$. Suppose that the theorem is not true. Then there exists a sequence of surfaces $M_i \in \mathfrak{M}(D, C, \chi, l, \epsilon)$ such that $L(M_i, N) \rightarrow \infty$, where N is a smooth surface of the same class. It will be shown later that we can suppose surfaces M_i to be equipped with polyhedral metrics.

Lemma 2 implies that there is a subsequence of $\{M_i\}$ converging in Gromov–Hausdorff topology and the limit space M for this subsequence is an Alexandrov space of the same class $\mathfrak{M}(\chi, D, C, l, \epsilon)$. In particular $\omega(p) \leq 2\pi - \epsilon$ for every point $p \in M$. Let us keep the same notation for this subsequence. From this and Theorem 1 from [BeBu], it follows that $L(M, N) < \infty$. Therefore we come to a contradiction if prove the following lemma.

Lemma 1 (Key Lemma). *Under the assumptions we made above, $L(M_i, M) \leq A < \infty$, where the constant A does not depend on i .*

The proof of this lemma is the main part of the proof of Theorem 1. It is exposed in Section 5. The proof is based on special triangulations of the surfaces M and M_i from Section 4 and on the basic construction of the paper [BL]. Auxiliary statements on triangles in \mathbb{R}^2 and Alexandrov surfaces are located in Section 3.

2 Space \mathfrak{M} is compact

Lemma 2. *The space $\mathfrak{M} = \mathfrak{M}(\chi, D, C, l, \epsilon)$ is compact in Gromov–Hausdorff topology.*

Proof. Precompactness of \mathfrak{M} was proved in [Sh]; we give here a short proof to make our exposition complete. Recall, that C^* means different constants depending on parameters of the class \mathfrak{M}

1. It is proved in [Sh] that the space \mathfrak{M} is precompact. Nevertheless we give a short proof here to do our text more self-contained. It is sufficient to show that for any (small enough) $r > 0$, on every surface $M \subset \mathfrak{M}$, there is a r -net containing not greater than C^*r^{-2} points.

Let us fix $r < \frac{1}{4}l$ and consider a maximal $2r$ -separated set $\{a_1, \dots, a_k\}$ of points of the surface M . These points form a $4r$ -net. Denote $\mathbf{D} = \mathbf{D}(a, r)$, where $a = a_i$ and let r_0 be the supremum of numbers $\rho \leq r$ such that the disk $\mathbf{D}(a, \rho')$ is simply connected for all $\rho' \leq \rho$.

If $r_0 \geq \frac{1}{2}r \sin \frac{\epsilon}{2}$, then $S(\mathbf{D}) \geq \frac{1}{8}\epsilon r^2 (\sin \frac{\epsilon}{2})^2$. As the whole area of M is not greater than $(2\pi + \omega^-(M)) \text{diam}^2 M \leq D^2(2\pi + C)$, the number of such disks is not bigger than C^*r^{-2} .

Now suppose that $r_0 \leq \frac{1}{2}r \sin \frac{\epsilon}{2}$. Then there is a geodesic loop $\mathbf{D}(a, r_0) \gamma$ of length $2r_0$ centered at a separating two components of the boundary of the disk $\mathbf{D}(a, r_0)$. As $2r < l$, at least one of components of $M \setminus \gamma$ being simply connected. Denote By K its closure. The Gauss–Bonnet formula says that $\omega^+(K) \geq \pi$. The well known inequality for length of a curve in a simply connected region (see, for example, [Resh], section 8.5) gives

$$R(K) \leq \frac{2r_0}{\sin \frac{\omega^+(K)}{2}} \leq \frac{2r_0}{\sin \frac{\epsilon}{2}} \leq r,$$

where $R(K)$ is inradius of K ; i.e.,

$$R(K) = \sup\{d(x, \partial K), x \in K\}.$$

This means that K does not intersect disks $\mathbf{D}(a_j, r)$, $j \neq i$. Besides, $\omega^+(K) \geq \pi$. If we add the set K to the disk \mathbf{D} , then we will get the set which does not intersect other disks and has curvature $\geq \pi$. After we perform the same for every disk with radius satisfying the condition $r_0 \geq \frac{1}{2}r \sin \frac{\epsilon}{2}$, we get a family of disjoint sets containing our disks. All different from disks sets have positive curvature at least π each. Therefore the number of such sets and the number of all disks can be estimated above by C^*r^{-2} .

2. It remains to prove that $M \in \mathfrak{M}$ if $M_i \in \mathfrak{M}$ and $M_i \rightarrow M$. In [Sh] it is proved that M looks like a graph (may be infinite) some vertices of which “are blown up” to Alexandrov surfaces; these surfaces can be glued together only along separate points, see details in [Sh]. Therefore it is sufficient to prove that every point $p \in M$ can not separate its neighborhood U . It becomes clear that $M \in \mathfrak{M}$ in this case. Indeed, obviously $\text{diam } M \leq D$. Curvatures ω_i of surfaces M_i converge weakly (in the sense of K. Fukaya’s definition, see

[Sh]) to curvature ω of M , therefore $\omega^-(M) \leq C$. Now it is easy to check that the condition (iii) from the definition of class \mathfrak{M} holds for M .

So let us prove that any point $p \in M$ can not separate its neighborhood. Reasoning to the contrary, suppose that there is a point $p \in M$ separating a some its neighborhood. Then it separates every its smaller neighborhood. Let p separate its round neighborhoods $U = \mathbf{D}(p, 10r) \supset \mathbf{D}(p, \rho) = \mathbf{D}$. Take points a, b in different components of $U \setminus p$, both at a distance r from p . Let us choose points $p_i, a_i, b_i \in M_i$ such that $p_i \xrightarrow{GH} p$, $a_i \xrightarrow{GH} a$, $b_i \xrightarrow{GH} b$ (we mean convergence in the sense of the Gromov–Hausdorff metric). For all sufficiently big i , distances $|a_i p|, |b_i p|$ are almost equal to r .

Now consider disks $U_i = \mathbf{D}(p_i, 10r)$, $\mathbf{D}_i = \mathbf{D}(p_i, \rho)$, where $\rho \ll r$, for instance $\rho < \frac{1}{100}(2\pi + C)^{-1}r$ and besides $r < \frac{1}{3}l$. Note that length of the disk $\mathbf{D}_i = \mathbf{D}(p_i, \rho)$ boundary is not greater than $(2\pi + C)\rho$. As $r < \frac{1}{3}l$, each closed disk $\bar{U}_i, \bar{\mathbf{D}}_i$ is homeomorphic to an Euclidean closed disk with not more than countable set of disjoint open disks removed.

Two cases are possible.

a) For some subsequence of indexes i , the points a_i and b_i are located in one component of $U_i \setminus \bar{\mathbf{D}}_i$. In this case points a_i, b_i can be connected by a path of the length not greater than $3r + (2\pi + C)\rho < 4r$ in $U_i \setminus \bar{\mathbf{D}}_i$. Replace this path by a dotted line with steps $\frac{1}{10}\rho$ having not bigger, than $40r\rho^{-1}$ points. Taking the limit, we get a dotted line whose steps are also small and which “connects” a and b in M . At least one of the points of this dotted line has to be not farther than $\frac{1}{10}\rho$ from p . This contradict to the fact that all distances between points of converging dotted lines and corresponding points p_i are not greater than ρ .

b) Let points a_i and b_i be in different components of the set $U_i \setminus \bar{\mathbf{D}}_i$ (for some subsequence). In particular, the closed disks $\bar{\mathbf{D}}_i$ are not simply connected. Then there is a simple closed loop in $\bar{\mathbf{D}}_i$ of length not greater than 3ρ such that it separates components containing the points a_i, b_i . This loop is contractible as $3\rho < l$. Therefore the loop bounds a disk \mathbf{D}' containing one of our components. Assume that just a_i are in this component. The Gauss–Bonnet theorem implies that $\omega^+(\mathbf{D}') \geq \pi$. Let us choose $\rho < \frac{1}{100} \sin \frac{\epsilon}{2} (2\pi + C)^{-1}r$. Then the distance from a_i to the boundary of \mathbf{D}' is not greater than

$$\frac{\text{boundary length of } \mathbf{D}'}{\sin \frac{\epsilon}{2}} \leq 3\rho(2\pi + C)(\sin \frac{\epsilon}{2})^{-1} \leq \frac{3}{100}r.$$

Hence, distances between points a_i and disks \mathbf{D}_i are not greater than $\frac{3}{100}r$. Thus, $|pa_i| \leq \rho + \frac{3}{100}r < \frac{1}{2}r$. Contradiction.

The lemma is proved.

3 Lemmas about triangles

Here we collect some auxiliary statements on triangles in Alexandrov surfaces. These lemmas will be used in Sections 5 and 6. Basically these lemmas are modifications of statements proved in [BeBu] and [AZ].

Along with usual triangles sometimes we will consider generalized triangles. By a generalized triangle, we mean a disk bounded by three broken lines (sides of the triangle) constructed from minimizers. It is supposed that lengths of these sides satisfy the strict triangle inequality. We call total curvature and denote by $\tilde{\Omega}(T)$ the sum of absolute curvature of a generalized triangle $T = \triangle ABC$ and variations of turn of its sides; i.e., $\tilde{\Omega}(T) = \Omega(T) + \sigma(AB) + \sigma(BC) + \sigma(CA)$, where σ means variation of turn from the triangle side. The angles of a generalized triangle are allowed to be zero. For short we will drop sometimes the word “generalized”.

Recall, that a comparison triangle for a (generalized) triangle T in Alexandrov space M is a planar triangle with the same side lengths.

Usually we will consider generalized triangles T for which $\tilde{\Omega}(T)$ is small enough. If this quantity is small in comparison with the angles of a triangle, then such a generalized triangle is bi-Lipschitz equivalent to its comparison triangle, where Lipschitz constant depends on low angles estimate. More precisely, the following statement takes place.

Lemma 3. *For any $\alpha > 0$, $L > 1$, there exists $\delta = \delta(\alpha, L) > 0$ with the following property. If every angle of a generalized triangle $\triangle ABC$ is not less than α and $\tilde{\Omega}(\triangle ABC) < \delta$, then there exists a L -bi-Lipschitz map of the generalized triangle $\triangle ABC$ onto its comparison triangle, this map may be chosen in such a way that its restriction on the boundary of the triangle is an isometry which moves every vertex to a vertex.*

This lemma is a minor modification of Lemma 4 from [BeBu] and can be proved by the same way. By this reason we drop the proof. We will also need a more general statement.

Lemma 4. *Let a simply connected closed region T is equipped with a polyhedral metric and bounded by two shortest curves BA , BC and a geodesic*

broken line AC . Suppose that $|BA| + |BC| > s(AC)$, where $s(AC)$ is the length of AC .

Assume that T is starlike with respect to a point C ; i.e., all shortest curves BX , where $X \in AC$, intersect AC at point X only. Let angle $\angle ABC$ satisfy the condition $0 < \phi \leq \angle ABC \leq \frac{1}{10}$. Also suppose that for every $X \in AC$, angles between a shortest BX and started at X arcs of the broken line AC are in the interval $[\frac{\pi}{2} - \frac{1}{10}, \frac{\pi}{2} + \frac{1}{10}]$.

Then there exist constants δ, L such that if $\tilde{\Omega}(T) < \delta$, then T is L -bi-Lipschitz equivalent to a planar triangle $\triangle A'B'C'$, whose side lengths are equal to $|AB|, |CB|, s(AC)$, correspondingly.

If in addition AC is a shortest curve, then one can choose L as a function $L = L(\delta)$ in such a way that $L \rightarrow 1$ as $\delta \rightarrow 0$.

Recall that we suppose that L -bi-Lipschitz map of T onto its “comparison triangle” $A'B'C'$ keeps lengths of boundary curves fixed.

This lemma also is a modification of Lemma 4 from [BeBu], and can be proved by the same way, so we omit details of the proof. The idea of the proof is the following. First of all we map T onto a planar closed region \tilde{T} bounded by intervals A_1B_1, C_1B_1 and a broken line A_1C_1 such that $|A_1B_1| = |AB|, |C_1B_1| = |CB|, \angle A_1B_1C_1 = \angle ABC$. To do this we use Tchebyshev coordinates. One can verify that the turns of the broken line A_1C_1 at its vertices can be estimated above by some value depending on ϕ, ϕ_1 and smallness of δ . After that, it is not difficult to map \tilde{T} onto the comparison triangle $\triangle A'B'C'$.

Besides we will need the following corollary.

Corollary 2. *Let a quadrangle $\square = AA_1C_1C$ be boundary convex and bounded by four shortest curves, $\tilde{\Omega}(\square) < \delta$. Suppose that*

$$|AA_1| = |CC_1|, \quad |AC| < \frac{1}{2}|AA_1|, \quad |A_1C_1| < \frac{1}{4}|AA_1|,$$

$$\angle A_1 < \frac{\pi}{2} - \phi, \quad \angle C_1 < \frac{\pi}{2} - \phi, \quad |\angle A - \frac{\pi}{2}| < \phi, \quad |\angle C - \frac{\pi}{2}| < \phi,$$

where $0 < \phi < \frac{1}{10}$. Then for every fixed ϕ , there is a function $L = L(\delta) \geq 1$ such that $L \rightarrow 1$ and $g \rightarrow 0$ as $\delta \rightarrow 0$ and \square is L -bi-Lipschitz equivalent to a planar quadrangle having the same side lengths and satisfying the condition: differences between its angles $\angle A'_1, \angle C'_1$ and angles $\angle A_1, \angle C_1$ are not greater than $C^*\delta$.

To prove let us separate the quadrangle \square from the surface and attach a planar triangle A_1C_1 along A_1C_1 such that its sides A_1O, C_1O are continuations of the quadrangle sides; i.e., they form angles π with the shortest

curves A_1A , C_1A_C , correspondingly. Thus, we obtain a generalized triangle $T = \triangle OAC$ (it is not necessary an ordinary triangle as its sides can be not shortest curves). It is not difficult to check that this triangle satisfies the conditions of Lemma 4. Applying this lemma gives a bi-Lipschitz (with a constant depending on ϕ and smallness of δ only) map $f_0: G \rightarrow \triangle O'A'C'$, where $\triangle O'A'C'$ is a comparison triangle for T ; restrictions of f_0 on the sides are isometries.

In the proof of Lemma 4, the map f_0 is constructed in two steps. First we map T onto a planar figure bounded by two intervals (the images of OA and OC) and the broken line γ (the image of the shortest curve AC). To do this we use Tchebyshev coordinates. As the second step, the broken line γ is transformed into an interval, see details in [BeBu].

As triangle $\triangle OA_1C_1$ is planar, the first map acts isometrically on it, in particular, the shortest curve A_1C_1 is mapped onto interval $A'_1C'_1$ of the same length. Now it is not difficult to straighten up the broken line γ keeping interval $A'_1C'_1$ fixed. To do this let us cut the quadrangle $A'A'_1C'_1C'$ by the diagonal A'_1C' into two triangles. Now we can straighten the broken line γ as a side of the “curved triangle” $\triangle A'_1C'A'$. For this we transform $\triangle A'_1C'A'$ the same way as it was done in the item 8 of the proof of Lemma 4 in [BeBu]. We keep the triangle $A'_1C'_1C'$ firm during this process.

Remark 1. The words “bi-Lipschitz equivalence” will always mean (if contrary is not supposed) the existence of a bi-Lipschitz map with a constant depending on parameters of the class \mathfrak{M} only. If a surface has the boundary, we suppose that the restriction of a bi-Lipschitz map on the boundary is linear. In case of triangles we also suppose that vertices are mapped into vertices.

The total curvature $\tilde{\Omega}(G)$ of a subset G of a generalized triangle $T = \triangle ABC$ is equal, by definition, to the sum of $\Omega(G)$ and negative turn of intersection of triangle sides with G (we mean open sides without vertices). Recall that turn of an ordinary triangle side is nonpositive.)

By shortest curves connecting points of a triangle we mean shortest curves of its induced metric.

Lemma 5. *For any positive Ψ , R , δ , there exists a number $r > 0$ having the following properties. Let a simple triangle $\triangle ABC$ satisfy the conditions: $\tilde{\Omega}(\triangle ABC) < \Psi$, $\tilde{\Omega}(\triangle ABC \setminus \mathbf{D}(Z, r)) < \delta$. Then,*

(i) if $B = Z$, $|AC| < R$, $d(B, [AC]) > R$, then the differences between angles $\angle B$, $\angle C$ of the triangle and corresponding angles of its comparison triangle $\triangle A'B'C'$ are not greater than 2δ .

(ii) If $A, B \in \mathbf{D}(Z, r)$, and $|CZ| \geq R$, then $\angle ACB - \angle A'C'B' \leq 2\delta$, where $\angle A'C'B'$ is the angle in the comparison triangle.

Remark 2. If we choose r such that $\angle A'C'B' < \delta$ in the item (ii), then obviously $\angle ACB < 3\delta$.

Here we restrict ourselves by a sketch of a proof, because technique of the proof is the same as in section 2 of chapter IV in the book [AZ]; the reader can find all details in the book. (Note, that it is enough to prove the lemma for polyhedral metrics only; by the way, we need only this case.)

In the item (i), the idea of the proof is the following: suppose that in our triangle (with a polyhedral metric), there are points of positive curvature at the distance less than $r < R/2$ from B . Then one can consecutively move these points X until they are placed at the distance at least $R/2$ from B . For this we look for a bigon (bounded by two shortest lines with common ends at B and one more point Y) containing point X and then remove the bigon. As a result, vertex X vanishes but additional curvature can appear at the point Y . This additional curvature at least $\frac{2r}{R}$ times less than curvature of the removed vertex X . This means that curvature of the vertex will be less than $\delta/2$ if $4\Psi r < \delta R$.

Now there is no positive curvature in $R/2$ -neighborhood of B . This allows to move all vertices X of negative curvature at the distance at least $R/2$ from B . To do this we glue an additional material in a slit looking like a tree with one vertex; it consists of BX and several additional slits started at X . At this step negative curvature decreases almost in the same proportion as positive curvature has been decreased. As a result variation of curvature becomes less than 2δ . The side AC keeps to be a shortest during this process because it was far enough of the deformed region of the triangle. Angles $\angle A$, $\angle C$ were not changed too. This proves the item (i).

In the item (ii) the idea of the proof is almost the same: at the first step we remove all vertices of positive curvature on the side AB by cutting bigons with vertex C . This allows to remove all positive curvature. Choosing r as in the item (i) we can guarantee that change of angle $\angle C$ is not greater than δ . However the side AB can cease to be a shortest curve. Let us replace it in such a case by a shortest curve (in the induced metric), which is not

longer. As a result, variation of curvature can only decrease. Applying the angle comparison theorem to the triangle of nonpositive curvature, bounded by AC , BC and a new shortest curve AB immediately gives the required inequality.

4 Approximations and triangulations

Lemma 6. *Every compact Alexandrov surface M (possibly with boundary) without peak points can be Lipschitz approximated by surfaces P_i with polyhedral metrics. Moreover, convergence $P_i \rightarrow M$ can be made regular; the latter means that $\omega_i^\pm \xrightarrow{\text{weak}} \omega^\pm$.*

This lemma was announced by Yu. Reshetnyak in [Resh1] (actually in a more general form), but the proof has never been published.

Proof. Recall that a triangle is simple if it is boundary convex, its sides have no common points except vertices and bound a disk. According to [AZ], Theorem 3 of Chapter 3, M can be partitioned onto arbitrary small simple triangles such that all triangle inequalities are strict. In addition, for any finite set of points and a finite set of shortest lines started at these points, it is possible to include these points to the set of vertices and some initial intervals of the shortest lines to the set of edges. Replacing each triangle of the partition by a planar triangle with the same side lengths (comparison triangle), we get a surface P equipped with a polyhedral metric. It is proved in [AZ], Theorem 7 of Chapter 7, that if triangles of the partitions become smaller and smaller, the sequence of polyhedra P_i converges to M uniformly and regularly.

Now we particularize our partition according to the purpose to provide Lipschitz convergence. Namely, let $\theta_0 = \frac{1}{100} \min_{p \in M} (2\pi - \omega(p))$. There is only a finite number of points with absolute curvatures greater than θ_0 . Denote them by E_1, \dots, E_m . We construct a partition such that the star of each point E_k , $k = 1, \dots, m$, consists of isosceles triangles with vertex E_k , angles of the triangles at E_k being in the interval $(2\theta_0, 10\theta_0)$. Besides, we do triangles of the partition so small that $\tilde{\Omega}(T) < 0,001\theta_0$ for every triangle T . As curvature of triangles is small, all the angles except may be one angle in any triangle to be less than $\pi - 5\theta_0$. After we cut each triangle with a “big” angle onto two triangles we get a partition such that all angles of triangles are less than $\pi - 5\theta_0$.

Now we change slightly our partition to get a partition all angles of which are positive. To do this we replace some ordinary triangles by generalized ones. We can do this in such a way that the stars of points E_k do not change and every changed side is transformed to a broken geodesic having almost the same length and turn as the replaced side (see details in Lemma 6 of the paper [BeBu]). This deformation is supposed to be so small that all the properties listed above are preserved.

Let i be so great that $\frac{1}{i} \ll \theta_0$. By $\delta_i = \delta(\theta_0, \frac{1}{i}) > 0$ denote the number corresponding to θ_0 and $L_i = 1 + \frac{1}{i}$ in according with Lemma 3. We can choose the partition of M onto generalized triangles T so that (in addition to properties mentioned above) the following holds: $\tilde{\Omega}(T) < 0,001 \min\{\delta_i, \theta_0\}$ and $\text{diam } T < \frac{1}{i}$.

By M^i we denote the surface M jointly with the partition we have chosen. All angles of (generalized) triangles T_{ij} of this partition are not zero and, therefore, they are not less than some number $\beta_i > 0$. The triangles T_{ij} having all angles not less than θ_0 are L_i -bi-Lipschitz equivalent to their comparison triangles (Lemma 3). In particular, it takes place for all triangles adjacent to vertices E_k .

Only one angle of any triangle T_{ij} can be less than θ_0 (because the triangle does not have “big” angles and its curvature is small). Let angle $\angle ABC$ of triangle $\triangle ABC$ be less than θ_0 and its other angles be greater than θ_0 . Obviously such a triangle can not be adjacent to vertices E_k . Take points A_1, C_1, B_1 on the sides AB, BC, AC so that $|AA_1| = |AB_1|$, $|CC_1| = |CB_1|$, $|BA_1| = |BC_1|$ (“Gromov’s product”). Let us connect these points with shortest lines in the induced metric of the triangle. Note that due to smallness of $\tilde{\Omega}(\triangle ABC)$, these shortest lines will cut $\triangle ABC$ onto 4 simple (generalized)triangles, all angles of these triangles, except may be $\angle A_1BC_1$, being greater than θ_0 . Now we choose points $A_2 \in A_1B, C_2 \in C_1B, B_2 \in A_1C_1$ such that $|A_1A_2| = |A_1B_2|$, $|C_1C_2| = |C_1B_2|$, $|BA_2| = |BC_2|$ and continue this process. It is not difficult to calculate that, as curvature is small, all angles of triangles $\triangle A_kA_{k+1}B_{k+1}, \triangle C_kC_{k+1}B_{k+1}, \triangle B_kA_kC_k$, are bounded below by θ_0 (we set $A = A_0, C = C_0, k = 0, 1, \dots$) and for sides of these triangles the strict triangle inequality holds. Hence, all these triangles are L_i -bi-Lipschitz equivalent to their comparison triangles. It is easy to see that $A_k \rightarrow B, B_k \rightarrow B$ as $k \rightarrow \infty$. Therefore there is a number k such that $\tilde{\Omega}(\triangle A_kBC_k) < \delta(\beta_i, L_i)$. This means that $\triangle A_kBC_k$ is L_i -bi-Lipschitz equivalent to its comparison triangle (Lemma 3). Now, replacing each triangle of our partition of $\triangle ABC$ by its comparison triangle (and, of

course, doing this for each triangle $\triangle ABC$) we obtain a polyhedron P_i , which is L_i -bi-Lipschitz equivalent to M .

Lemma 6 is proved.

Lemma 7. *For every $\nu > 0$, $d > 0$, each compact (possibly with boundary) Alexandrov surface M without peak points has a triangulation $\{T_k\}$ such that*

- (i) $\tilde{\Omega}(T_k) < \nu$;
- (ii) $\text{diam } T_k < d$;
- (iii) *all angles of triangles T_k are not less than $\alpha(\theta) > 0$ where α depends on $\theta = \min\{\min\{2\pi - \omega^+(p) : p \in M\}, \min\{\pi - \tau(q) : q \in \partial M\}\}$ only. Here $\tau(q)$ is turn at point q .*
- (iv) *The set of vertices contains any given a priori finite set of points $E_k \in M$.*

Remark 3. a) In case the metric of M is polyhedral, this lemma was in fact proved in [B] (Theorem 2) on the basis of the theorem from [BZ] (see also [BZ1]); all triangles of the triangulation are flat in this special case.

b) Probably, using Tchebyshev coordinate, it is possible to prove Lemma 7 the same way as it has been proved for polyhedra in [B], [BZ]. However it is simpler to reduce Lemma 7 to the case of polyhedra with the help of Lemma 6.

Proof. From Lemma 6 it follows that M can be Lipschitz approximated by polyhedra P_i . Let $f: P_i \rightarrow M$ be corresponding L_i -bi-Lipschitz maps, $L_i \rightarrow 1$ as $i \rightarrow \infty$. Fix a set $\{F_k\}$ in M . We include all the points with variation of curvature greater than $\frac{1}{10}\nu$ to this set. Denote $F_{ki} = f_i^{-1}(F_k)$. As it was mentioned, P_i can be triangulated onto planar triangles satisfying conditions (i)-(iv) of the lemma, even if we replace numbers ν , d to $\frac{1}{100}\nu$, $\frac{1}{10}d$ beforehand. Choosing such a triangulation of P_i we can include all points F_{ki} to the set of vertices. Also we can suppose the triangles to be so small that every d -neighborhood of each point $A \in M$ contains not more than one point F_k and absolute curvature of such a neighborhood without point F_k is not greater than $\frac{1}{20}\nu$. Also we can suppose that the similar is true for every polyhedron P_i if i is big enough. One can choose the described triangulation of the polyhedron P_i in such a way that all angles of the triangles are bounded below by some number 2α depending on $\theta_i = \min\{\min\{2\pi - \omega^+(p) : p \in P_i\}, \min\{\tau(q) : q \in \partial P_i\}\}$ only; in particular, 2α does not depend on smallness of triangles. (Note, that numbers θ_i for

polyhedra P_i with great i are almost the same as the corresponding number θ for M .) Let us set α to be equal a half of this number. Now we use Lemma 3. As α does not depend on smallness of triangles, the choice of points (F_k) and numbers ν, d , we can assume ν to be so small in comparison with α that $2\nu < \delta = \delta(\alpha, L = 2)$, where δ is defined by Lemma 3.

Now connect by shortest curves points of M , whose inverse images in P_i are connected by shortest curves (keeping i fixed). We claim that, if i is big enough, this makes a triangulation of M combinatorially equivalent to the triangulation of P_i , all angles of this triangulation being separated from zero by a number depending on θ only and the angles at F_k being only slightly (less than 2ν) different from corresponding angles at F_k .

Indeed, let AB and BC be the edges of the triangulation of P_i , $A'B'C'$ the shortest curves in M , correspondingly. The shortest curves AB BC divide a neighborhood of B onto two sectors. The sector corresponding to triangle ABC is distinctly smaller and its angle is equal to the angle $\angle ABC$ of the triangle. In addition $\angle A'B'C'$ is almost equal to $\angle ABC$ if i is big. Combinatorial equivalence of the nets follows easily from this. Other properties of the triangulation of M now follow from corresponding properties of triangulations of polyhedra P_i (if i is sufficiently big).

Lemma 7 is proved.

5 Proof of Key Lemma

1. *Preliminary agreements.* Here we will consider only a sequence of surfaces $M_j \in \mathfrak{M}$ converging (in Gromov–Hausdorff topology) to a surface M . We will construct partitions of these surfaces into triangles. These triangles we suppose to be so small that the values of arguments χ, D, l of class \mathfrak{M} do not play any role in our consideration. By bi-Lipschitz equivalence of triangles or more general figures, we *always* mean a bi-Lipschitz map with a constant depending on C and ϵ only. If there are marked points in the boundary of a figure (we claim that vertices of a triangle are always marked), we assume that our map moves marked points to marked ones and that the restriction of the map on boundary curves connecting marked points is linear.

2. *Choice of scales.* We have three scales. First, it is the size of angles of triangles. Partitions of the limit surface M are constructed of two types triangles: “ordinary” and “special” ones. In accordance with Lemma 7, angles of ordinary triangles are separated from zero by some constant $\lambda > 0$

depending on C and ϵ only. All special triangles are isosceles, and angles at their vertices belong to the interval (φ_0, φ_1) where φ_i are small positive numbers also depending on C and ϵ only; they will be chosen in item 3 of the proof.

At the second step we choose a positive number δ to be so small that conclusions of Lemma 3 and Lemma 4 holds even if angles of triangles are bounded below by the number $0,01\pi\varphi_0(2\pi + C)^{-1}$ instead of φ_0 . Some quantities such that they can be estimated above by $C^*\delta$, where C^* depends on C and ϵ only, will arise in the process of the proof. By δ' we denote such quantities. It is important that we can unboundedly decrease δ and, therefore, δ' keeping C and ϵ fixed. By this reason we will drop a factor m in quantities of the form $m\delta$ if m is not too big (say, less than 50). It is convenient to assume that $\delta' \ll \min\{\varphi_0, \lambda\}$.

After we have fixed φ_i and δ we choose a partition of M into so small triangles that variation of curvature for every triangle is less than $< \delta$. (By variation of curvature for a triangle T we mean $\tilde{\Omega}(T)$.) In fact we choose the partition even more petty. This helps us to transfer the partition to the surfaces M_j for big values of j .

Finally, fixing a partition, we choose so great integer j_0 , that for $j > j_0$ essential portions of curvature of M_j are concentrated in very small (in comparison with size of the triangles) neighborhoods of vertices.

Let us explain the last point. K. Fukaya defined weak convergence of measures for the case of Gromov–Hausdorff convergence of spaces, see details in [Sh]. For a subsequence curvatures ω_j of M_j converge weakly to curvature ω of M ; positive and negative parts ω_j^+ , ω_j^- of ω converge weakly to some finite measures μ^+ , μ^- . We have $\mu^\pm \geq \omega^\pm$, where ω^\pm are positive and negative parts of ω . Choosing a partition of M onto triangles we require that not only variation of curvature but also measures μ^+ , μ^- be small (less than δ) on all triangles with vertices removed. (Note, that both measures, μ^+ and μ^- , can be big simultaneously at a vertex. The reason is that the convergence $M_j \rightarrow M$ can be nonregular. All vertices for which these measures are big are special.) However for converging surfaces M_j , measures ω^\pm are not necessary concentrated at vertices, they can be “spread out”. Hopefully, for any $R > 0$ there exists a number j_0 such that, for any vertex B of a special triangle T , almost all $\omega^\pm(T)$ are concentrated in R -neighborhood $V = \mathbf{D}(B, R)$ of point B_j for $j > j_0$.

This means that for every special triangle $\triangle A_j B_j C_j = T$

$$\omega_j^\pm(T \setminus V) < \delta. \quad (1)$$

later on we suppose j to be so big that the inequality (1) holds true for R we have chosen.

3. *Special vertices and triangles.* Let us consider the limit surface M . Its partition will be based on Lemma 7. Before applying the lemma we choose a finite set of points $F_k \in M$ and triangulate small closed neighborhoods Q_k^0 of these points in a special way. We suppose that $Q_k^0 \cap Q_l^0 = \emptyset$ for $k \neq l$. After that, we apply our Lemma 7 to the surface $M_0 = M \setminus \cup Q_k^0$ with boundary. As a result we obtain a partition of M onto triangles.

We set Q_k^0 to be stars of points F_k . These stars consist of isosceles triangles $\triangle F_k A_{ki} A_{k i+1}$, where $|F_k A_{ki}| = |F_k A_{k i+1}|$. We call points F_k and triangles $\triangle F_k A_{ki} A_{k i+1}$ adjacent to them to be *special*. Construction of these stars has some freedom; in particular angles of the special triangles, their size and pettiness of triangulation can be changed. We will use this freedom as follows.

Let C and ϵ be constants from the definition of class \mathfrak{M} , $C_1 = 2\pi + C$. First we choose intervals for values of the angles with vertices at F_k (before choosing points F_k). These angles should be so small that even being multiplied by $2\pi/\epsilon$ they remain “small”, say, less than 0,001. From the other hand we should bound uniformly these angles below and bound a number of edges at a special vertex above. So we require that these angles ψ to be in the interval

$$\varphi_0 = 10^{-5}\epsilon < \psi < \varphi_1 = 10^{-4}\epsilon. \quad (2)$$

These conditions are always met in such a way that the number m of edges at F_k is uniformly bounded above:

$$m < 10^6(2\pi + C)\epsilon^{-1}. \quad (3)$$

Now we choose the number δ , which characterizes smallness of curvature of triangles. Namely, set $L = \frac{11}{10}$ and let δ_1 be a number corresponding to the numbers L and $\alpha = \lambda$ in according with Lemma 3. Similarly we can find δ_2 , corresponding to $L = 0,01\varphi_0 C_1^{-1}$. Then Lemma 4 gives us δ_3 , corresponding to $\phi = 0,1\varphi_0$, $\phi_1 = 0,01$. Finally we put $\delta = \frac{1}{100} \min_i \{\delta_i\}$. Hence, δ depends on C , ϵ only. Recall that we can decrease δ if we need and after that find a partition of M onto triangles such that absolute curvatures of the triangles

do not exceed the new value of δ ; low bounds of triangle angles will not be changed. As $\varphi_1 < 0,01$, we can suppose that adjacent to the base angles of special triangles are close to $\pi/2$ (up to φ_0).

After we fix set $\{F_k\}$ (we will do that some later) we will choose stars Q_k^0 of these vertices to be so small that $\Omega(Q_k^0 \setminus F_k) < \delta$ and besides $\text{diam } Q_k^0 < \delta$. Hence, each special triangle will be $\frac{11}{10}$ -bi-Lipschitz equivalent to its comparison triangle. Note that turn of the boundary of Q_k^0 from outside at any point is not big, say, less than $\pi/2$.

4. *Partition of M onto triangles.* Let us triangulate the surface $M' = M \setminus \cup_k Q_k^0$ in according with Lemma 7. All angles of such a triangulation are bounded below by some number $\lambda > 0$ depending on the number θ of M_0 (see item (iii) of Lemma 7). The last number actually does not depend on our choice of vertices F_k and their stars, so we can set $\theta = \epsilon$. Indeed, as it was mentioned above, outside turn of the boundary of any star at any point is not greater than $\frac{1}{2}\pi$. At the same time, including all points having big values of μ^\pm in the set $\{F_k\}$, and taking a sufficient petty triangulation, we can provide the inequality $\tilde{\Omega}(T) < \delta$ for all triangles, with δ as chosen above.

Thus, from the beginning we include all the points having curvature $\Omega(F_k) \geq \delta$ in F_k ; after that we choose stars Q_k to be so small that $\Omega(Q_k \setminus F_k) \leq \delta$; and finally we triangulate M_0 so that for any triangle the inequality $\tilde{\Omega}T \leq \delta$ holds. This is possible, as our constants do not depend on the choice of the set of points F_k , stars Q_k and a triangulation. In fact, we will add some requirements (which can easily be fulfilled) on the choice of partition of M in the beginning of item 5.

As a result we get a partition of M onto two kinds of triangles: special ones and others, each triangle T satisfying $\tilde{\Omega}(T) < \delta$ and being $\frac{11}{10}$ -bi-Lipschitz equivalent to its comparison triangle.

5. *Converging surfaces.* Lemma 6 allows us to think that converging surfaces M_j are equipped with polyhedral metrics. Taking a subsequence, we can suppose that curvatures ω_j of surfaces M_j converge weakly (in the sense of definition from [Sh]) to curvature of M , their positive and negative parts ω_j^+ , ω_j^- converge weakly to some finite measures μ^+ , μ^- . Recall that $\mu^\pm \geq \omega^\pm$, where ω^\pm are positive and negative parts of curvature of M .

Consider the partition of M chosen in the item 4 of the proof. Let $\{A_i\}$ be the set of all the vertices of the partition, $\{F_k\}$ be its subset consisting of the special vertices. Taking more reach set $\{F_k\}$, small stars Q_k and making triangles smaller, we can include all points $X \in M$ with $\mu^\pm(X) \geq \delta$ in set

$\{F_k\}$ and ensure every closed triangle with vertices removed to satisfy the inequality $\mu^\pm < \delta$. The condition (iii) from definition of classes \mathfrak{M} implies $\mu^+(X) \leq 2\pi - \epsilon$ for every point $X \in M$. As a result, we can ensure all triangles to be so small that the inequality $\mu^+(Q_k^0) < 2\pi - \frac{2}{3}\epsilon$ holds for each star.

Let diameters of all triangles are not greater than a number $d > 0$ so small that

- (a) the inequality $\mu^\pm(E) < \delta$ holds for every set E such that it does not contain points F_k and its diameter $\text{diam}(E) \leq 10d$;
- (b) each circle of radius $10d$ contains not more than one vertex F_k .

Denote by A_{jk} points of the surface M_j such that $A_{jk} \xrightarrow{GH} A_k$ as $j \rightarrow \infty$; in particular, $F_{jk} \xrightarrow{GH} F_k$. For a vertex A_k belonging to the boundary of a star Q_l^0 , let us choose points A_{jk} so that $|F_{jl}A_{jk}| = |F_lA_k|$.

Later on we assume numbers j to be so big that if a set $B \subset M_j$ has diameter $\leq 6d$ and does not intersect δ -neighborhoods of points F_{jk} , then $\Omega_j(B) = \omega_j^+(B) + \omega_j^-(B) < \delta$

6. *Partitions of surfaces M_j and non-special triangles.* To construct a partition of the surfaces M_j , connect pairs of points A_{jk} by shortest curves if and only if corresponding pairs of points A_k are connected by shortest curves. Such shortest curves are not necessary unique and can have superfluous intersections one with another. We will choose shortest curves in a way to avoid such extra intersections. Note, that shortest curves connecting A_{jk} with A_{js} are not necessary converge (in Gromov–Hausdorff metric sense) to shortest paths between A_k A_s chosen beforehand. Almost the same arguments as in Lemma 7 show that we get a partition combinatorial equivalent to the partition of the surface M .

Let $\triangle ABC$ of the surface M be non-special. Its angles are almost the same as angles of its comparison triangle. If numbers j are great enough, triangles $\triangle A_j B_j C_j$ are in regions with small variation of curvature (less than δ). Hence, the angles of such a triangle are almost equal to the angles of its comparison triangle. Lemma 4 from [BeBu] implies that both triangles, $\triangle ABC$ and $\triangle A_j B_j C_j$, are bi-Lipschitz equivalent to their comparison triangles with a constant L depending on λ and δ only (in notations of the lemma). This constant can be chosen as close to 1 as we wish, if δ is small enough. For a great j both comparison triangles, $\triangle ABC$ $\triangle A_j D_j C_j$, are almost equal. So, all non-special triangles of the surfaces M_j are bi-Lipschitz equivalent to corresponding triangles of the surface M .

Therefore, to finish the proof, it is sufficient to verify that (for great j) every special triangle of the surface M_j is bi-Lipschitz equivalent to the corresponding triangle of the surface M or, equivalently, to its comparison triangle.

7. *Special triangles.* Let Q_0 be the star of a fixed vertex $B^0 = E_k$ of the surfaces M , Q be the star of the corresponding vertex $B = E_{kj}$ of the surfaces M_j . Recall that triangles of Q_0 are almost flat, so that they are bi-Lipschitz equivalent to their comparison triangles; the latter being bi-Lipschitz equivalent to comparison triangles for corresponding triangles of Q . This shows that it is sufficient to prove that (for sufficiently great j) every star Q is bi-Lipschitz equivalent to the star glued from comparison triangles for the triangles of Q .

8. *Plan of further proof.* We are going to apply to Q arguments from [BL]. To do this we attach a plane with a disk removed to Q and so we obtain a complete surface P homeomorphic to the plane. Recall that the key part of the proof in [BL] is, roughly speaking, the following statement. If P is a polyhedral surface homeomorphic to the plane, $\omega^+(P) \leq 2\pi - \epsilon < 2\pi$, and $\omega^-(P) \leq C < \infty$, then there is a set of flat sectors with disjoint interiors on P ; every point of nonzero curvature being a vertex for some sectors. We can decrease or increase (depending on the sign of curvature) these sectors so that curvature at the sector vertices vanishes. Size of sector angles implies that this process comes to a bi-Lipschitz map with a constant L depending on C and ϵ only. So we obtain a bi-Lipschitz map of P to the plane \mathbb{R}^2 .

Actually such a transformation requires three steps in [BL]. First P is divided into two half planes by a special quasi-geodesic, and the flat sectors are chosen separately in every half plane. After that the vertices of positive curvature are removed. Finally vertices of negative curvature are removed. See details in [BL].

There is an obstacle for direct application of this construction in our case. It is flat sectors containing rays that form small angles with the boundary $\Gamma = \partial Q$ of star Q . Sectors on P with vertices close to Γ can have such a property. To avoid this difficulty, we choose j_0 so great that almost all curvature of Q is concentrated in a very small neighborhood V of the central point $B \in Q$ for $j > j_0$. After that, we replace a wide collar of ∂Q by a flat collar in Q . As a result, flat sectors come out to be almost orthogonal to ∂Q on the new deformed surface. This simplifies further considerations.

9. *Elimination of curvature near ∂Q .* Let $\{A_i\}$ be the set of vertices of

∂Q , $|BA_i| = R_0$. Put

$$\kappa = 10 \max\left\{\frac{2\pi}{2\pi - \epsilon}, \frac{2\pi + C}{2\pi}\right\}. \quad (4)$$

Consider the disk $\mathbf{D}(B, R)$ of radius R such that

$$10\kappa R < \delta R_0. \quad (5)$$

After that, we choose disk $\mathbf{D}(B, r)$ (where $r \ll R$) and great number j_0 such that $\mu^\pm(Q \setminus \mathbf{D}(B, r)) < \delta$ for $j > j_0$ and, besides, the conditions of Lemma 5 hold for $\Psi = C$, $Z = B$.

We are going to show that every star Q is bi-Lipschitz equivalent to a region Q' (equipped with a polyhedral metric) which flat everywhere except a C^*r -neighborhood of a point Z' located at a distance C^*R_0 from the boundary of Q' ; $\mu^-(Q') < C + \delta$ and $\mu^+(Q') < 2\pi - \frac{1}{2}\epsilon$.

To simplify notations we omit indices and denote by $\triangle BAC$ triangle $\triangle BA_iA_{i+1}$. Take points A_1, C_1 on the shortest curves BA, BC at distance $r/2$ from B and connect these points by a shortest curve A_1C_1 .

Let us strengthen our requirement about j_0 ; namely, choose $\rho > 0$ so small and j_0 so great that conditions of Lemma 5 hold even if we replace R and r to r and ρ , correspondingly. In particular, we have $\mu^\pm(Q \setminus \mathbf{D}(B, \rho)) < \delta$. It is not difficult to see that in this case A_1C_1 is contained in $\mathbf{D}(B, r)$ and can not visit not only the disk $\mathbf{D}(B, \rho)$, but even the disk $\mathbf{D}(B, r/4)$, and angles $\angle BA_1C_1$ and $\angle BC_1A_1$ are “almost equal” to angles $\angle A'_1, \angle C'_1$ of comparison triangle $\triangle B'A'_1C'_1$ (i.e., their differences are not greater than δ). In particular, these angles are less than $\frac{1}{2}(\pi - \varphi_0)$. It is easy to see that the conditions of Corollary 2 hold for the quadrangle AA_1C_1C (with an appropriate ϕ). Let us apply the corollary. This allows us to replace each triangle $\triangle BA_iA_{i+1}$ by a triangle flat outside the disk $\mathbf{D}(B, r)$ and L -bi-Lipschitz equivalent to $\triangle BA_iA_{i+1}$. Even if variation of curvature of the new triangle is greater than variation of curvature of the old triangles (at points A_1, C_1), change of curvature is not greater than $C^*\delta$. If we choose sufficiently small δ and sufficiently great j_0 , we can take constant L as close to 1 as we wish.

Let us save old notations $Q, B, A_1A_2 \dots A_m$ for a new star arranged from the new triangles and elements of the star.

Besides, we suppose r to be so small in comparison with R , that $\triangle ABC$ satisfies the conditions $A \in \mathbf{D}(B, \kappa r)$, $|BC| \geq \kappa^{-1}R$.

10. *Flat sectors.* We want to prove that a new star Q is bi-Lipschitz equivalent to a star obtained by gluing together comparison triangles for

triangles of the star Q . To do this, we apply the construction from [BL], described above in short, in item 8. This construction has to be applied twice: first, to remove positive curvature and, after that, to remove negative one. These two steps are similar, so we will consider in details only the first one.

Let us supply Q with a flat annulus to obtain an open complete surface P , flat everywhere except the disk $\mathbf{D}(B, r) \subset Q$. This is possible. Indeed, denote by α_i^- and α_i^+ adjacent to the base angles of triangle $\triangle A_i B A_{i+1}$. Consider a flat region bounded by two rays and interval of the length $|A_i A_{i+1}|$ under condition that angles between the interval and the rays from the region side are equal to $\pi - \alpha_i^-$, $\pi - \alpha_i^+$, correspondingly. Glue these flat regions together along rays and attach the obtained region to Q . For j great enough, the surface P satisfies the conditions: its positive curvature is less than $2\pi - \epsilon - \delta = 2\pi - \epsilon'$ and negative one is less than $C + \delta = C'$. Since our estimates are rough, we preserve for ϵ' and C' previous notations ϵ and C .

It follows from [BL] that there exists a finite set of flat sectors with disjoint interiors on P such that all vertices of sectors are just vertices of positive curvature and the sum of angles for sectors with a common vertex O equals

$$\frac{2\pi - \omega^+(P)}{\omega^+(P)} \omega^+(O).$$

To remove positive curvature at the point O , we stretch all sectors with vertex O by increasing their angles in $L_1 = \frac{2\pi}{2\pi - \omega^+(P)}$ times. As a result, we obtain a polyhedron P_1 of nonpositive curvature bi-Lipschitz equivalent to P .

After this step, one finds an analogous system of flat sectors with vertices at points of negative curvature and removes negative curvature in the same way by means of compressing flat sectors of P_1 . Finally we have a bi-Lipschitz map $f: P_1 \rightarrow \mathbb{R}^2$ with the Lipschitz constant $\left(\frac{2\pi+C}{\epsilon}\right)^{\frac{1}{2}}$.

Following [BL], we use maps of the form $(r, \phi) \rightarrow (r, a\phi)$ for stretching and compressing sectors, (r, ϕ) being polar coordinates with origin at the vertex of a sector. We can assume that angles of the sectors are not big, in particular, that each sector intersect only one special triangle base and the central point B of the star Q does not belong to the interior of a sector. To achieve this, it is enough to divide sectors onto smaller ones.

Actually we will consider not all surfaces P , but only stars Q of points B . Such a star consists of isosceles triangles $\triangle A_i B_j A_{i+i}$ and is bounded by the

geodesic broken $\Gamma = A_1 A_2 \dots A_m$. From description of the map f , it is clear that Q is bi-Lipschitz equivalent to a flat region — its image $\tilde{Q} = f(Q)$. The map f transforms bases $\Gamma_i = A_i A_{i+1}$ of triangles $\triangle A_i B_j A_{i+i}$ to curves $\tilde{\Gamma}_i$ (not smooth in general). These curves consist of straight segments (images of segments which do not belong to a flat sector) and smooth curves (images of intersection of Γ_i with a flat sector). (It is not essential for us how images of lateral sides of triangles $\triangle A_i B_j A_{i+i}$ look like.)

11. *Flat region \tilde{Q} .* We are going to show that the flat region \tilde{Q} is bi-Lipschitz equivalent to a polygon glued from comparison triangles for curved triangles of Q .

Let us connect \tilde{B} with points \tilde{A}_i by shortest curves in intrinsic metric of \tilde{Q} (avoiding unnecessary intersections). So we divide \tilde{Q} onto “curved triangles” \tilde{T}_i with curves $\tilde{\Gamma}_i$ as bases. (It will be clear later that these shortest curves are almost orthogonal to \tilde{Q} and do not touch one another.)

It is sufficient to verify that

(a) every curved triangle \tilde{T}_i is bi-Lipschitz equivalent to its comparison triangle (i.e, a flat triangle \tilde{T}'_i with side lengths equal to $|\tilde{B}\tilde{A}_i|$, $|\tilde{B}\tilde{A}_{i+1}|$, and $s(\tilde{\Gamma}_i)$, correspondingly);

(b) the last flat triangle is bi-Lipschitz equivalent to a comparison triangle for $\triangle B A_i A_{i+1}$.

As the last triangle is almost equal (for great j) to the corresponding triangle of the star Q^0 , this ends the proof.

To prove (a) and (b), we need to estimate the angle and the distance distortions for map f . To simplify exposition, we will consider only one step (removing positive curvature); the estimates for the second step (removing negative curvature) are analogous.

12. *Estimates.* To prove (a), we use Lemma 4. The following statements show that \tilde{T}_i satisfies the conditions of this lemma. Also they help us to prove (b).

(i) For j great enough, map f slightly changes distances from B to boundary Γ of Q . More precisely, for every $X \in \partial Q$ the inequality

$$||\tilde{B}\tilde{X}| - |BX|| < C^* \delta |BX| \quad (6)$$

holds.

(ii) Let $X \in \tilde{\Gamma}_i$; then angles between radial shortest curves $\tilde{B}\tilde{X}$ and arcs of $\tilde{\Gamma}_i$ starting at X are close to $\frac{1}{2}\pi$. In particular, turns of $\tilde{\Gamma}_i$ at its angular

points are small. Words “close” and “small” mean that difference between angles $\pi/2$ is not greater than $\frac{1}{10}$.

Proof (i). Let $X \in \partial Q$. Prove that $|BX| < (1 + \delta)|\tilde{B}\tilde{X}|$. The second required inequality is proved by analogy.

Consider a shortest curve $\tilde{\alpha}$ connecting \tilde{B} with $\tilde{X} \in \partial\tilde{Q}$ and its f -inverse image α . From (5) it follows that the initial arc $\tilde{\alpha}_0$ of $\tilde{\alpha}$, from \tilde{B} to the boundary of f -image of $\mathbf{D}(B, R)$, is not longer than $\kappa R < \frac{1}{10}\delta R_0 \leq \frac{1}{5}\delta|BX|$.

If a segment of the shortest curve α does not visit flat sectors, map f does not change its length. If a segment of α is outside the disk $\mathbf{D}(\tilde{B}, 2r_2)$ and contained in a flat sector which was constricted, it could become only shorter under f .

Now let $\tilde{\beta}$ be an interval of $\tilde{\alpha}$ containing in a flat sector \tilde{S} such that f got stretched S , and β f -inverse image in S of the shortest path $\tilde{\beta}$. Let O, \tilde{O} be vertices of sectors S, \tilde{S} , correspondingly.

Denote by Y, Z the initial and the end points of segment β , and by \tilde{Y}, \tilde{Z} the initial and the end points of $\tilde{\beta}$. If Z belongs to ∂Q , we replace sectors S, \tilde{S} by their subsectors $ZOY, \tilde{Z}\tilde{O}\tilde{Y}$, and preserve previous notations S, \tilde{S} for the new sectors.

We can suppose that $\tilde{\beta}$ does not intersect the initial segment $\tilde{\alpha}_0$, so the distance between \tilde{B} and \tilde{Y} is not less than R , and therefore (see Lemma 5, item (ii) and Remark 3) $\angle\tilde{O}\tilde{Y}\tilde{B} \leq 3\delta$.

Let us show that

$$s(\tilde{\beta}) \leq (1 + C^*\delta)s(\beta), \quad (7)$$

where, as usual, C^* means a constant depending on C, ϵ only.

Denote $\angle YOZ = \phi, \angle\tilde{Y}\tilde{O}\tilde{Z} = \tilde{\phi}, |OZ| = |\tilde{O}\tilde{Z}| = b, |YZ| = c, |\tilde{Y}\tilde{Z}| = \tilde{c}, \pi - \angle OYZ = \chi$.

Place triangles $\triangle OYZ$ and $\triangle\tilde{S}\tilde{Y}\tilde{Z}$ in \mathbb{R}^2 to one half-plane with respect to their common side $OY = \tilde{O}\tilde{Y}$. Now it is clear that $|\tilde{c} - c| \leq |\tilde{Z}Z| = 2\sin\frac{1}{2}(\tilde{\phi} - \phi)b \leq (\kappa - 1)\delta c$, as $\phi < \chi \leq \delta$. The last inequality follows from our choice of disk $\mathbf{D}(B, r)$ in the beginning of item 9 and from Lemma 5. So the estimate (7) is proved.

To obtain the second estimate it is enough to take the shortest curve BX and its f -image in capacity of α and $\tilde{\alpha}$ correspondingly.

Proof (ii). We start with consideration of Q and, to be short, denote $A_i = A, A_{i+1} = C$. Side AC is small in comparison with $|AB| = |CB|$ (see inequalities (2)). Therefore adjacent to base $A'C'$ angles of comparison

triangle $\triangle A'B'C'$ are close to $\pi/2$ and angle $\angle A'B'C'$ is small. The item (i) of Lemma 5 says that angles $\angle BAC$, $\angle BCA$ are close to $\pi/2$ either.

Consider a triangle $\triangle BAX$, where $X \in AC$. Let $\triangle B'A'X'$ be its comparison triangle. Again from the item (i) of Lemma 5, it follows that angles $\angle A$, $\angle X$ are equal correspondingly to angles $\angle A'$, $\angle X'$. As $\angle A'$ is almost equal $\pi/2$, $\angle A$ is close to $\pi/2$ too. ("Close" means that their difference has the order of $0,01\epsilon + \delta$.) Taking into account that angle $\angle A'B'X'$ is small, from this it follows that angle $\angle A'X'B'$ is also close to $\pi/2$. Now, again by item (i) of Lemma 5, angle $\angle AXB$ is close to $\pi/2$ too. The same is true for angle $\angle CXB$.

Let S be a flat sector with a vertex O , sides of the sector intersect AC at points X, Y . Point O is in the small neighborhood $\mathbf{D}(B, r)$ of B , but not necessary in the triangle $\triangle ABC$. It follows from Lemma 5, item (ii) that the angles $\angle OXB$, $\angle OYB$ are small; therefore the angles $\angle OXY$, $\angle OYX$ are close to $\pi/2$ (by the same scale: their difference has the order $0,01\epsilon + \delta$).

Now pass to sector \tilde{S} , the image of flat sector S . Radii of flat sector S are almost orthogonal to Γ_i . A straightforward calculation shows that from this it follows that radii of flat sector \tilde{S} are almost orthogonal to $\tilde{\Gamma}_i$. Distinction of the last angles from $\pi/2$ depends on distinction between angles Γ_i and radii of flat sectors S from $\pi/2$ and on κ ; i.e., finally on C and ϵ only.

Vertices O of flat sectors S are very close to B . Dilatation of f is not greater than κ , so f -images of vertices O are close to \tilde{B} . Hence, $\angle OXB$, where $X \in \Gamma_i$, are close to zero, so angles between segments $\tilde{B}\tilde{X}$ (they are shortest curves in \tilde{Q}) and Γ_i are close to $\pi/2$. (In particular, flat region \tilde{Q} is a star region with respect to \tilde{B} .) This proves item (ii).

Estimate (i) implies that differences between length of the sides $\tilde{A}\tilde{B}'$, $\tilde{C}'\tilde{B}'$ of the comparison triangle $\triangle \tilde{A}\tilde{B}'\tilde{C}'$ and length of the sides of the comparison triangle $\triangle A'B'C'$ are small. From (ii) it follows that ratio of $|\tilde{A}\tilde{C}'|$ to $|\tilde{A}'\tilde{C}'|$ is bounded from below and above by numbers depending on C and ϵ only. The choice of φ and item (ii) imply that angle $\angle \tilde{A}\tilde{B}'\tilde{C}'$ is less than $\pi/2$. From this it becomes clear that flat triangles $\triangle \tilde{A}\tilde{B}'\tilde{C}'$ and $\triangle A'B'C'$ are L -bi-Lipschitz equivalent, where L depends on C and ϵ only; for example, see Corollary 1 in the paper [BeBu]. Finally, each triangle $\triangle \tilde{A}\tilde{B}\tilde{C}$ is bi-Lipschitz equivalent to the corresponding triangle of the star Q^0 , and our theorem is proved completely.

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